Admissions Testing Service

STEP Solutions 2015

Mathematics

STEP 9465/9470/9475

October 2015

Test

STEP 2 2015 Hints and Solutions

Question 1

For the first result, show that the gradient of the function is positive for all positive values of x (by differentiating) and also that $f(0) \geq 0$. Once this result has been established sum a set of the terms, using $x = \frac{1}{k}$, note that $\ln\left(1 + \frac{1}{k}\right)$ can be written as $\ln(k+1) - \ln(k)$ and then the required result follows.

For the second part, first show that $x + \ln(1 - x)$ is *negative* for $0 < x < 1$ and then use the substitution $x = \frac{1}{k^2}$, noting that $\ln\left(1 - \frac{1}{k^2}\right)$ can be written as $\ln(k-1) - 2\ln(k) + \ln(k+1)$. Deal with the sum starting with $k = 2$ and then add the initial 1 afterwards.

Question 2

As with all geometric questions a good diagram of the information given makes the solution to this question much easier to reach. The first result in this question follows from an application of the sine rule with applications of the relevant formulae for $\sin(A + B)$ and the double angle formulae. From a diagram of the triangle it should then be an easy application of trigonometry to show that $DE = \frac{x}{2}$. There are a number of different methods for establishing that FC trisects the angle ACB – one method is to show that $\sin(\angle FCE) = \frac{1}{2}$, following which it is relatively straightforward to work out the sizes of angles ACB and ACF in terms of α and show that they must satisfy the correct relationship.

Question 3

For the first part note that $T_8 - T_7$ can be interpreted as the triangles that can be made using the rod of length 8 and two other, shorter rods. These can then be counted by noting that there are 6 possibilities if the length 7 rod is used, 4 possibilities if the length 6 (but not the length 7) rod is used and 2 possibilities if the length 5 (but not 6 or 7) rod is used. It is clear that at least one rod longer than length 4 must be used. To evaluate $T_8 - T_6$ note that it is equal to $(T_8 - T_7) + (T_7 - T_6)$ and then evaluate $T_7 - T_6$ in a similar manner to $T_8 - T_7$. Similar reasoning easily gives formulae for $T_{2m} - T_{2m-1}$ and $T_{2m} - T_{2m-2}$.

For the induction, the rule for $T_{2m} - T_{2m-2}$ deduced in the previous part can be used to show the inductive step, while the easiest way to show the base case is to list the possibilities. The easiest way to establish the result for an odd number of rods is to use the formula for $T_{2m} - T_{2m-1}$ and the formula for T_{2m} that was just proven.

Question 4

For the first part, note that the graph of $\arctan x$ satisfies the requirement of being continuous, but does not satisfy $f(0) = \pi$. Since $\tan(x + \pi) = \tan x$, a translation of the graph of $y = \arctan x$ vertically by a distance of π gives the required graph.

It should be clear that the graph of $y = \frac{x}{1+x^2}$ has no vertical asymptotes, approaches the x-axis as $x \to \pm\infty$ and passes through the origin. Identifying the stationary points should be the next task after which a graph should be easy to sketch. The graph of $y = g(x)$ should then be easy to sketch by considering the fact that $f(x)$ is an increasing function and $g(x)$ is obtained by composing the two functions already sketched.

To sketch the graph of $y = \frac{x}{1-x^2}$ first note that there must be two vertical asymptotes. Once stationary points have been checked for it should be straightforward to complete the sketch. In this case, the asymptotes need to be considered to deduce the shape of the graph for $y = h(x)$ as the composition with $f(x)$ will lead to discontinuities. Noting again that $\tan(x + \pi) = \tan x$ the discontinuities can be resolved by translating sections of that graph vertically by a distance of π.

Question 5

The initial proof by induction is a straightforward application of the $\tan(A + B)$ formula. The final part of section (i) requires recognition that there are many possible values of x to give a particular value of $\tan x$, but only one of them is the value that would be obtained by applying the arctan function. The result can therefore be shown by establishing that the difference between consecutive terms of the sequence is never more than π.

For the second part of the question a diagram of the triangle and application of the $\tan 2A$ formula shows that the value of α_n must be of the form used in the first part of the question. All that remains is then to show that the limit of the sum must give the required value.

Question 6

The first part of the question requires use of the $\cos(A + B)$ formula. Following this the integral should be easy to evaluate given that $\int \sec^2 x \, dx = \tan x + c$. In the second part, apply the substitution and note that the limits of the integral are reversed, which is equivalent to multiplying by -1. Following this a simple rearrangement (noting that the variable that the integration is taken over can be changed from y to x) should establish the required result. The integral at the end of this part can then be evaluated simply by applying this result along with the integral evaluated in part (i).

In the final part of the question it is tempting to make repeated applications of the result proven in part (ii). However, this is not valid as it would require the use of a function satisfying $f(\sin x) = x$, which is not possible on the interval over which the integral is defined. Instead, application of a similar substitution to part (ii) to $\int_0^\pi x^3 f(\sin x)\, dx$ will simplify to allow this integral to be evaluated based on the integration of $\frac{1}{(1+\sin x)^2}$. An application of the result from part (ii) will also be required.

Question 7

For part (i) note that the lines joining the centres of the two circles and one of the points where the bisection occurs form a right-angled triangle, so the radius of the new circle can be calculated. To show that no such circle can exist when $r < a$ note that the diametrically opposite points on C must be a distance of $2a$ apart, and no two points on a circle of radius r can be that far apart. For the case $r = a$ note that the new circle would be the same as C (and so would have more than two intersection points).

For part (ii) a similar method can be used to deduce the distances between the centre of the new circle and each of C_1 and C_2. From these distances equations can be formed relating the x and y coordinates of the centre of the new circle. It is then an easy task to eliminate the y-coordinate of the centre of the circle from the equations to get the given value of the x-coordinate.

The expression for y can easily be found by substituting back into the equations obtained from the distance between the centres of two of the circles. Once this is done, note that $y^2 \geq 0$ to obtain the final inequality.

Question 8

The first part of the question follows from consideration of similar triangles in the diagram if the line through P and the centres of the circles is added. For the second part, expressions can be written down for the position vectors of Q and R by noting that the same method as in part (i) will still apply. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ can then be compared to show that one is a multiple of the other.

For the final part of the question, note that Q will lie halfway between P and R if $\overrightarrow{PQ} = \overrightarrow{QR}$.

Question 9

A diagram to represent this situation will show the angles that will be required to calculate the moments of each of the particles about A in terms of θ. Following this, simple trigonometric manipulation should lead to a relationship between $\sin\theta$ and $\cos\theta$. From this, either a right-angled triangle or one of the basic trigonometric identities can be used to reach the required result.

For the second part of the question the amount of potential energy that needs to be gained by the system should be easy to calculate and this must be equal to the initial kinetic energy of the system.

Question 10

The component of the velocity of the particle in the direction of the string at any moment must be equal to V, which leads to $V \operatorname{cosec} \theta$ as the speed of the particle along the floor. Alternatively, introduce a variable to represent the length of string still in the room or the height of the room and then differentiate x, the distance of the particle from the point directly beneath the hole, with respect to time. The length of the string (to the hole in the ceiling) is decreasing at a rate of $V\ ms^{-1}$, which then allows the introduced variable to be eliminated to reach an expression for the speed of the particle.

Differentiation of the speed of the particle allows the acceleration to be calculated. Finally, note that the particle will remain on the floor as long as the vertical component of the tension is less than the weight of the particle and then the point at which the particle leaves the floor can be identified.

Question 11

For the first part, the coordinates of A are found by applying simple trigonometric ratios and differentiation with respect to time gives the velocity of A. In the second part, the first equation results from consideration of conservation of momentum and the second results from conservation of energy (with a substitution based on the first equation made to eliminate one variable).

Since no energy is lost in any collisions the relationships from part (ii) must continue to hold and this shows that $\dot{\theta}$ cannot be 0 which means that the direction in which θ changes remains the same unless there is a collision. Since the first collision occurs when $\theta = 0$, the second one must be when $\theta = \pi$.

For the final part, note that the equations in part (ii) must still hold, and if $v = 0$, the kinetic energy of B must be 0. Since the kinetic energies of A and C must be equal (by symmetry) it must be the case that the kinetic energy of A is $\frac{1}{4}mu^2$ and can also be calculated from the expression for the velocity of A shown in part (i). Since $\dot{\theta}^2 > 0$, this can then be used to find the values of θ. Finally, note that given these values of θ, v will only be 0 on the occasions when $\dot{\theta}$ is positive.

Question 12

For the first part, note that A can only win the game if the first two tosses result in heads, since once there has been a tail, B will win as soon as two consecutive heads have been tossed and A cannot win until there have been two consecutive heads and one further toss. In the second part, note that this logic still applies to the game for A and similar reasoning can be applied to the game for C. For the other two players switching heads and tails in any sequence that results in a win for B will give a sequence that results in a win for D, and vice versa, so the probabilities must be equal. Since only sequences which alternate between heads and tails forever (and the probabilities of such sequences tend to zero as the lengths of the sequences increase) the probabilities must both also be ¼ .

For the final part, note that C must win if the first two tosses are TT. Since only the previous two tosses are important in determining what could happen on the next toss, each case can be analysed by a tree diagram which shows the outcomes after one further toss.

For example, following HT:

- H gives the position if the last two tosses were TH, and so a probability of winning of q,
- T gives the position if the last two tosses were TT and so a probability of winning of 1.

The total probability is therefore $\frac{1}{2}q + \frac{1}{2}$, but this must also be equal to p.

This yields three equations in the three unknowns which allows all of the individual probabilities to be calculated. Once this is done the overall probability can be calculated.

Question 13

To calculate the expected value of the total cost, note that there is a constant component of ky and then the expected value of the $a(X - y)$ given that $X > y$ must be added, which can be calculated by integration of $(x - y)\lambda e^{-\lambda x}$ with respect to x, between y and ∞. Differentiating the expression for $E(C)$ with respect to y allows the position of the stationary point to be found. If this is at a negative value then y should be chosen to be 0 and otherwise the value of y for the stationary point should be used.

A slightly more complicated integration is needed to establish the formula for $Var(C)$ and then differentiation of this gives a value that is clearly negative for positive values of y, which shows that the variance is decreasing as y increases.